THE DYNAMICS OF 2-GENERATOR SUBGROUPS OF PSL(2, C)

Robert Brooks* and J. Peter Matelski

A classical result of Shimizu and Leutbecher (see, for instance [6], p. 59) asserts that if $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ generate a discrete subgroup of $PSL(2,\mathbf{C})$, then either $c = 0$ or $|c| \geq 1$. This has been strengthened by T. Jørgensen [4] as follows:

JØRGENSEN'S INEQUALITY. If X and Y generate a discrete, non-elementary subgroup of $PSL(2,\mathbf{C})$, then

$$|\mathrm{tr}^2(X)-4| + |\mathrm{tr}(XYX^{-1}Y^{-1})-2| \geq 1 .$$

In this paper, we will show the existence of a sequence of inequalities, generalizing Jørgensen's inequality, which X and Y must satisfy in order for $\langle X, Y \rangle$, the group generated by X and Y, to be discrete. These conditions are mutually independent in the sense that, for given X and Y, at most one can fail to hold. These conditions arise from the Shimizu-Leutbecher process defined below.

For convenience, consider the upper half space model of hyperbolic 3-space. We denote a directed geodesic ℓ by the ordered pair of its end-points; so $\ell = (a, b)$, $a, b \in \mathbf{C}$, $a \neq b$. The *complex distance* $\tau = \delta(\ell_1, \ell_2)$

*Partially supported by NSF Grant #MCS 7802679.

Riemann Surfaces and Related Topics
Proceedings of the 1978 Stony Brook Conference
0-691-08264-2/80/000065-07 $00.50/1 (cloth)
0-691-08267-7/80/000065-07 $00.50/1 (paperback)
For copying information, see copyright page

between two directed geodesics $\ell_1 = (a_1, b_1)$ and $\ell_2 = (a_2, b_2)$ is defined as follows: $\tau \in \mathbf{C}$; $\mathrm{Re}(\tau) \geq 0$ is the hyperbolic distance between the geodesics; $\mathrm{Im}(\tau)$ is the angle made by the geodesics along their common perpendicular and is determined modulo 2π unless $\mathrm{Re}(\tau) = 0$, in which case $\pm \mathrm{Im}(\tau)$ is determined modulo 2π. One may compute the complex distance by the formula:

$$\cosh^2(\tau/2) = (a_1, a_2, b_2, b_1) ,$$

where (z_1, z_2, z_3, z_4) is the usual cross ratio, as can be checked if $\ell_1 = (-1, 1)$ and $\ell_2 = (-e^\tau, e^\tau)$.

Let X be a loxodromic element of $\mathrm{PSL}(2, \mathbf{C})$ and axis(X) the directed geodesic in hyperbolic space joining the fixed points of X. If ℓ is a perpendicular to axis(X), then the complex distance τ between ℓ and $X(\ell)$ is called the *complex translation length* of X. In fact X translates $\mathrm{Re}(\tau)$ units along axis(X) and rotates hyperbolic space by $\mathrm{Im}(\tau)$ about axis(X). We have

$$\mathrm{tr}^2 X = 4\cosh^2(\tau/2) ,$$

which makes sense even if X is not loxodromic.

Given X loxodromic with complex translation length τ, and Y in $\mathrm{PSL}(2, \mathbf{C})$, one may check the formula:

$$\mathrm{tr}((YXY^{-1})X^{-1}) - 2 = -(1 - \cosh(\tau))(1 - \cosh(\beta)) ,$$

for β the complex distance from axis(X) to axis(YXY^{-1}); this follows by normalizing

$$X = \begin{pmatrix} \cosh(\tau/2) & \sinh(\tau/2) \\ \sinh(\tau/2) & \cosh(\tau/2) \end{pmatrix}$$

$$YXY^{-1} = \begin{pmatrix} \cosh(\tau/2) & e^{\beta}\sinh(\tau/2) \\ e^{-\beta}\sinh(\tau/2) & \cosh(\tau/2) \end{pmatrix}.$$

Given X and Y elements of PSL(2, C) with X loxodromic, we define the Shimizu-Leutbecher sequence inductively by:

$$Y_1 = YXY^{-1}, \quad Y_{i+1} = Y_i X Y_i^{-1}.$$

Let τ be the complex translation length of X, and let β_i be the complex distance between axis(X) and axis(Y_i). A necessary condition for the group generated by X and Y to be discrete is that the set $\{\cosh(\beta_i)\}$ should form a discrete subset of $\mathbf{C}$.

The following lemma allows one to compute $\cosh(\beta_i)$ inductively:

LEMMA. $\cosh(\beta_{i+1}) = (1-\cosh(\tau))\cosh^2(\beta_i) + \cosh(\tau)$.

This follows from the hyperbolic law of cosines: if ℓ_0, ℓ_1, ℓ_2 are given, the law of cosines gives a formula for $\omega = \delta(\ell_1, \ell_2)$ in terms of $\tau_1 = \delta(\ell_0, \ell_1)$, $\tau_2 = \delta(\ell_0, \ell_2)$ and a which is the complex distance from the perpendicular between ℓ_0 and ℓ_1 to the perpendicular between ℓ_0 and ℓ_2. The formula is:

$$\cosh(\omega) = \cosh(\tau_1)\cosh(\tau_2) - \cosh(a)\sinh(\tau_1)\sinh(\tau_2).$$

The lemma follows by setting $\tau_1 = \tau_2 = \beta_i$ and $a = \tau$. One way to check the law of cosines is to normalize so that $\ell_0 = (0, \infty)$, $\ell_1 = (t_1, t_1^{-1})$, and $\ell_2 = (et_2, et_2^{-1})$ where $t_1 = \tanh(\tau_1/2)$, $t_2 = \tanh(\tau_2/2)$, and $e = e^{a}$; then compute $\cosh^2(\omega/2) = (t_1, et_2, et_2^{-1}, t_1^{-1})$. Note that ℓ_2 does indeed have complex distance τ_2 to ℓ_0 with $(-e^{a}, e^{a})$ as common perpendicular.

Now let $z_i = (1-\cosh(\tau))(\cosh(\beta_i))$. We may rewrite the above inductive formula as:

$$z_{i+1} = z_i^2 + C, \quad \text{where} \quad C = (1-\cosh(\tau))(\cosh \tau),$$

and we have that if X and Y generate a discrete group, then $\{z_i\}$ forms a discrete subset of $\mathbf{C}$.

The dynamical behavior of $\mathbf{C}$ under a quadratic polynomial is well understood from the work of Fatou-Julia ([1], [5]; see also [2]). Let

$f^i(z) = \underbrace{f \circ f \cdots \circ f}_{i \text{ times}}(z)$, where $f(z) = z^2 + C$; a solution ρ of the polynomial equation $f^i(z) = z$ will be called a stable periodic point of period i if $\left|\frac{d}{dz} f^i(\rho)\right| < 1$. Then f^i is contracting on any disk $B_\varepsilon = \{z : |z-\rho| < \varepsilon\}$ on which $\left|\frac{d}{dz} f^i\right| < 1$. The theorem of Fatou-Julia ensures that, for any choice of C, there is at most one stable periodic orbit. Further results of Fatou-Julia allow one to draw by computer the region E of $\mathbb{C}$ defined by $E = \{z : f^i(z)$ converges to the stable periodic orbit$\}$ (see Fig. 1), and the region of $\mathbb{C}$ defined by $\{C : z^2 + C$ has a stable periodic orbit$\}$ (see Fig. 2).

To obtain the above-mentioned inequalities, let p be a stable periodic point of f of period n; we may assume that $|p| < 1/2$. Expanding $f^n(z) = \sum_{i=0}^{2^n} a_i(z-p)^i$ as a Taylor series about p, we have

$$|f^n(z)-p| = |z-p| \left|\sum_{i=1}^{2^n} a_i(z-p)^{i-1}\right| \le |z-p|(2^n-1)m\,[\max(1, |z-p|^{2^n-1}]$$

where $m = \max(|a_i|)$. Setting $K \le \dfrac{1 - \left|\frac{d}{dz} f^n(p)\right|}{(2^n-1)m}$, we see that on the disk $|z-p| < \min(K, 1)$, f^n is a contracting map. If also $K < \dfrac{\left|\frac{d}{dz} f^n(p)\right|}{m \cdot (2^n-1)}$, then $f^n(z) - p$ has no roots other than p in the disk $|z-p| < K$.

In the case $n = 1$, the fixed points of $f(z) = z^2 + (1-\cosh(\tau))(\cosh(\tau))$ are $\cosh(\tau)$, $1-\cosh(\tau)$. If $|1-\cosh(\tau)| < \frac{1}{2}$, we may set $p \le 1-\cosh(\tau)$, and $\frac{df}{dz}(p) = 2(1-\cosh(\tau))$, $m = 1$. We thus have the inequality: If $0 < |(1-\cosh(\tau))(\cosh(\beta)-1)| < \min(1-2|\cosh(\tau)-1|,\ 2|\cosh(\tau)-1|)$, then $\langle X, Y \rangle$ is not discrete. In view of our expressions for $\cosh(\tau)$ and $\cosh(\beta)$ given above, this becomes Jørgensen's inequality.

In the case $n = 2$, the periodic points of order 2 of $f(z) = z^2 + C$ are $\dfrac{-1 \pm \sqrt{1-4(C+1)}}{2}$, and $\frac{d}{dz} f^2(p) = 4(C+1)$. Using the estimates

$|p| < \frac{1}{2}$, $|C| < \frac{5}{4}$, we find $m \leq 4$, so that $0 < |z-p| < \frac{1}{12}\min(1-4|C+1|, 4|C+1|)$ implies $<X, Y>$ is not discrete.

In terms of the matrix functions of X and Y, this may be rewritten as:

$$0 < \left| \mathrm{tr}(XYX^{-1}Y^{-1}) - \frac{\mathrm{tr}^2(X)}{2} - \frac{-1 \pm \sqrt{(\mathrm{tr}^2(X) - 5(\mathrm{tr}^2(X)-1)}}{2} \right|$$

$$< \frac{1}{12}\min\left[(1 - |\mathrm{tr}^4(X) - 6\mathrm{tr}^2(X) - 4|, |\mathrm{tr}^4(X) - 6\mathrm{tr}^2(X) + 4|)\right]$$

implies $<X, Y>$ is not discrete.

ACKNOWLEDGEMENTS: We would like to thank Henry Laufer for suggesting and assisting us with the use of the computer, and Bernard Maskit for his advice and encouragement. Also, we want to thank Dusa McDuff for showing us Fatou's paper.

REFERENCES

[1] P. Fatou, *Sur les Equations Fonctionelles*, Bull. Soc. Math. France, 48, (1920), 33-94.

[2] F. Gross, *Factorization of Meromorphic Functions*, Mathematics Research Center, Naval Research Laboratory, Washington, D. C., 1972.

[3] T. Jørgensen, *A Note on Subgroups of SL(2, $\mathbb{C}$)*, Quart. J. Math. Oxford (2), 28 (1977), 209-212.

[4] ————, *On Discrete Groups of Möbius Transformations*, Amer. J. Math., Vol. 98, No. 3, 739-749.

[5] G. Julia, *Memoire Sur l'Iteration des Fonctions Rationelles*, J. Math. 8, 1 (1918).

[6] I. Kra, *Automorphic Forms and Kleinian Groups*, W. A. Benjamin, Reading, Mass., 1972.

[7] J. P. Matelski, *A Compactness Theorem for Fuchsian Groups of the Second Kind*, Duke Math. J. Vol. 43, No. 4, (1976), 829-840.

Fig. 1. The set E for $f(z) = z^2 + C$ with $C = .1 + .6i$.

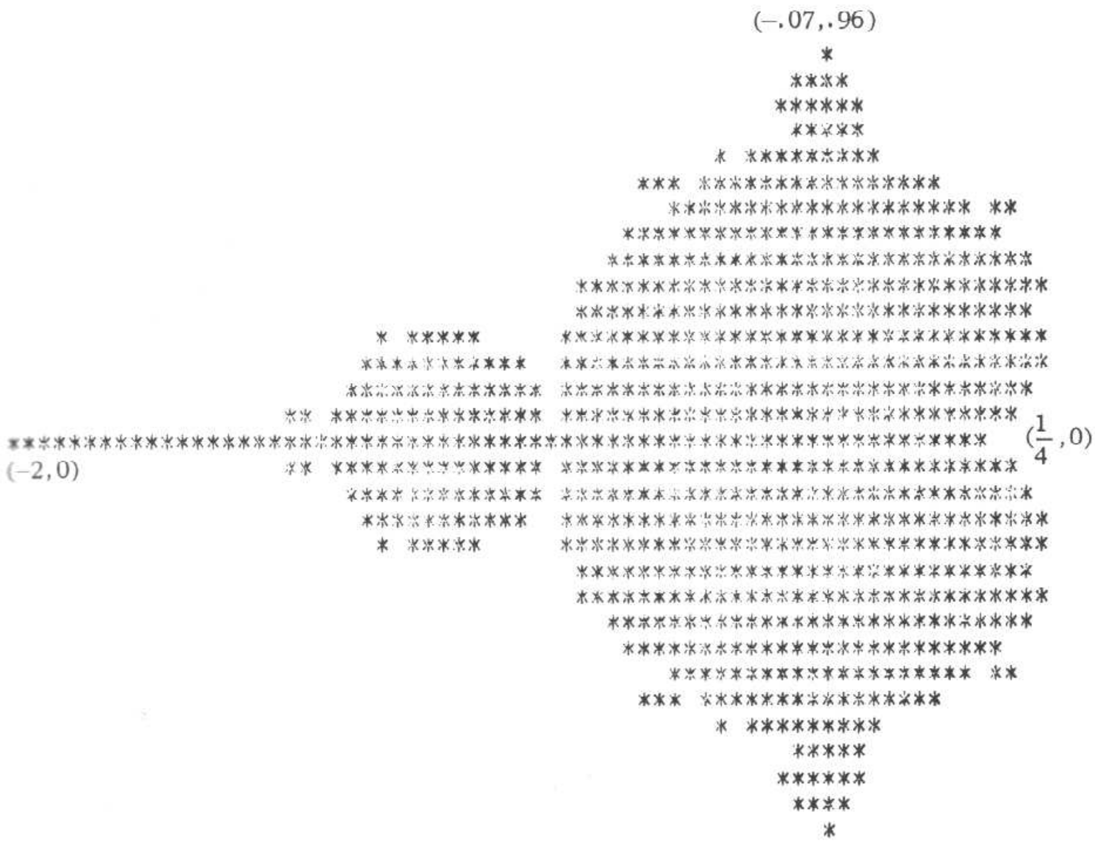

Fig. 2. The set of C's such that $f(z) = z^2 + C$ has a stable periodic orbit.

RIEMANN SURFACES AND RELATED TOPICS: PROCEEDINGS OF THE 1978 STONY BROOK CONFERENCE

EDITED BY

IRWIN KRA

AND

BERNARD MASKIT

PRINCETON UNIVERSITY PRESS

AND

UNIVERSITY OF TOKYO PRESS

PRINCETON, NEW JERSEY

1981